Quantum noncommutative gravity in two dimensions

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February 7, 2008

Abstract

We study quantisation of noncommutative gravity theories in two dimensions (with noncommutativity defined by the Moyal star product). We show that in the case of noncommutative Jackiw-Teitelboim gravity the path integral over gravitational degrees of freedom can be performed exactly even in the presence of a matter field. In the matter sector, we study possible choices of the operators describing quantum fluctuations and define their basic properties (e.g., the Lichnerowicz formula). Then we evaluate two leading terms in the heat kernel expansion, calculate the conformal anomaly and the Polyakov action (as an expansion in the conformal field).
PACS: 04.60.-m, 04.62.+v

1 Introduction

Over recent years noncommutative field theory has developed into a mature discipline (see reviews [1]). However, formulation of a satisfactory noncommutative counterpart of quantum gravity still remains an open problem. There exist several approaches to noncommutative gravity. One of them studies deformations of geometrical structures (as, e.g., differential structures and exterior algebras). An overview of this approach with the emphasis on two-dimensional models and further references can be found in [2]. Although this approach is very efficient for finding deformations of particular geometries, it does not refer to any action functional. Therefore, it is unclear how one could proceed with quantisation of such models.

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In noncommutative theories the action functional can be constructed by using the spectral action principle [3] which relates the action to the heat trace asymptotics of a suitable Dirac type operator. However, in many interesting cases (as, e.g., gravity on the Moyal plane) such an operator is not known so far (cf. [4, 5]).

In this paper we are interested in gravity theories where noncommutativity appears, roughly speaking, due to the presence of the Moyal star product (see eq. (1) below). A very fruitful approach to such theories is based on the gauge theory formulation of general relativity. In the noncommutative case the Lorentz group does not close and one has to deal with an extended gauge symmetry [6, 7]. Here we are interested in two-dimensional gravity. Therefore, the 2D model constructed in [8] is of particular importance for us. This model shares some similarities with the noncommutative 3D Chern-Simons gravity [9]. We also like to mention four-dimensional gauge gravity models [10, 11] and perturbative calculations in the Einstein gravity [12, 13] and of graviton scattering on a D-brane [14].

Dilaton gravities in two dimensions (see [15] for a recent review) have always been a good testing ground for various theoretical ideas and methods of classical and quantum general relativity. In particular, it was demonstrated [16] that in many cases the path integral for such models can be calculated exactly. We would like to check whether this property can be found in the noncommutative case as well. So far, the Jackiw-Teitelboim (JT) model [17] is the only two-dimensional dilaton gravity model which has a noncommutative counterpart [8]. Therefore, our study of the path integral is restricted to the noncommutative JT gravity. We show that in the temporal gauge, which tremendously simplifies the analysis, one can indeed perform the path integration over all gravity variables exactly and nonperturbatively (provided quantum matter interactions with gravity satisfy some mild restrictions). As a result, the full effective action becomes just a sum of the classical action in the gravity sector and of an effective action for the mater field calculated as if the gravity fields were a fixed background.

In the second part of the work we deal with the matter effective action (so that the restriction to the JT model becomes inessential). We use another important property of two-dimensional theories which we would like to keep in the noncommutative case. Quantum effective action for a matter field minimally coupled to gravity is uniquely defined by the conformal anomaly and, therefore, can be calculated exactly (giving the famous Polyakov action [18]). In the noncommutative case the order in which we multiply fields becomes essential. Therefore, even classical analysis of the coupling of matter fields to gravity becomes very complicated on the combinatorial side. To reduce this complexity we work in the conformal gauge. The restriction to the conformal gauge does not allow us to analyse symmetries of the model, so that this problem is postponed. However, we show that in the conformal gauge the requirements of hermiticity and of proper commutative limit are strong enough fix the fluctuation operators for spinors and scalars almost uniquely. As a byproduct we derive a Moyal extension of the Lich-

nerowicz formula. In the scalar case we derive then the heat kernel expansion, calculate the conformal anomaly, and integrate it to obtain a noncommutative version of the Polyakov action. Explicit formulae are given as an expansion in the conformal field.

This paper is organised as follows. In the next section we review some basic properties of the noncommutative JT gravity [8]. Section 3 is devoted to the path integral over the gravitational degrees of freedom. Effective action for the matter fields is studied in section 4. Section 5 contains a short overview of solved and unsolved problems. Some useful formulae can be found in Appendix A and B.

2 Noncommutative Jackiw-Teitelboim gravity

The Moyal product of two functions f and g on $\mathbb{R}^2$ can be defined by the equation

$$f \star g = f(x) \exp\left(\frac{i}{2}\,\theta^{\mu\nu}\overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu\right) g(x)\,. \tag{1}$$

θ is a constant antisymmetric matrix. In this form the star product has to be applied to plane waves and then extended to all (square integrable) functions by means of the Fourier series. This product is known for a long time in the operator theory (cf. [19]) and in deformations of symplectic manifolds [20].

A noncommutative deformation of the Jackiw-Teitelboim model has been constructed in [8]. It has been identified with a $U(1,1)$ gauge theory on noncommutative $\mathbb{R}^2$ with the action

$$S = \int \mathrm{tr}\,(\Phi \star F)\ , \tag{2}$$

where both fields Φ and F take values in the Lie algebra $u(1,1)$ of $U(1,1)$. The field Φ is a space-time scalar, and F is a two-form field strength with the components

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu \star A_\nu - A_\nu \star A_\mu\,. \tag{3}$$

The action (2) is invariant under usual (noncommutative) $U_\star(1,1)$ gauge transformations

$$A \to g_\star^{-1} \star A \star g_\star + g_\star^{-1} \star dg_\star, \qquad \Phi \to g_\star^{-1} \star \Phi \star g_\star\,. \tag{4}$$

Next one expands A and Φ over a basis τ_i in the defining 2-dimensional representation of $u(1,1)$: $A = \tau_i A^i$, $\Phi = \tau_i \Phi^i$. Precise form of this basis can be found in [8]. Then one introduces new fields according to the equations

$$\Phi^i = (l\phi^a, \phi, \psi), \qquad A^i_\mu = (e^a_\mu l^{-1}, \omega_\mu, b_\mu)\,, \tag{5}$$

where $a = 0, 1$, and the scale l is related to the cosmological constant $\Lambda = -1/l^2$. e^a_μ plays the role of the zweibein, $\varepsilon^a{}_b\omega_\mu + i\delta^a_b b_\mu$ is identified with an $so(1,1) \oplus u(1)$

connection. In terms of these new fields the action (2) reads

$$S = \frac{1}{4} \int d^2x \, \varepsilon^{\mu\nu} \left[\phi_{ab} \star \left(R^{ab}_{\mu\nu} - 2\Lambda e^a_\mu \star e^b_\nu \right) - 2\phi_a \star T^a_{\mu\nu} \right] \tag{6}$$

with the curvature tensor

$$\begin{aligned} R^{ab}_{\mu\nu} = \varepsilon^{ab} & \left(\partial_\mu \omega_\nu - \partial_\nu \omega_\mu + \frac{i}{2}[\omega_\mu, b_\nu] + \frac{i}{2}[b_\mu, \omega_\nu] \right) \\ & + \eta^{ab} \left(i\partial_\mu b_\nu - i\partial_\nu b_\mu + \frac{1}{2}[\omega_\mu, \omega_\nu] - \frac{1}{2}[b_\mu, b_\nu] \right) \end{aligned} \tag{7}$$

and with the noncommutative torsion

$$\begin{aligned} T^a_{\mu\nu} = \partial_\mu e^a_\nu - \partial_\nu e^a_\mu + \frac{1}{2} \varepsilon^a{}_b \left(\{\omega_\mu, e^b_\nu\} - \{\omega_\nu, e^b_\mu\} \right) \\ + \frac{i}{2} \left([b_\mu, e^a_\nu] - [b_\nu, e^a_\mu] \right) \, . \end{aligned} \tag{8}$$

The fields ϕ and ψ are combined into

$$\phi_{ab} := \phi \varepsilon_{ab} - i\eta_{ab}\psi \, . \tag{9}$$

All commutators (denoted by square brackets) and anticommutators (denoted by curved brackets) are calculated with the Moyal star product. Further conventions and notations can be found in Appendix A.

The fields Φ^i play the role of the Lagrange multipliers. ϕ_{ab} are responsible for a two-dimensional noncommutative version of the Einstein equations. Variation of (6) with respect to ϕ_a gives the torsion constraint

$$\varepsilon^{\mu\nu} T^a_{\mu\nu} = 0 \, . \tag{10}$$

The gauge transformations (4) can be rewritten in terms of the component fields (5) (see Appendix B). The four-parameter symmetry group contains Lorentz boosts, translations (which coincide with the diffeomorphisms on shell [8]), and additional $U(1)$ gauge transformations which are needed to close the gauge group in the noncommutative case. Note that the noncommutativity parameter θ is not changed under these transformations.

If $\theta = 0$ the fields b_μ and ψ decouple, and the dynamics of the rest of the fields is described by the commutative JT model.

All classical solutions of the noncommutative JT model have been found in [8]. In the present paper we deal with quantum theory only.

3 Exact path integral

It was demonstrated in [16] that in the JT model the path integral over the gravitational degrees of freedom can be performed exactly even in the presence

of matter fields. Here we extend this result to the noncommutative case. An important ingredient is a convenient gauge choice which simplifies the calculations enormously. Note, that the technique we employ here is very general. It has been used in other dilaton gravity models [21, 22] and in two-dimensional supergravities [23]. As a practical application of this approach we may mention calculations of loop corrections to the specific heat of the dilaton black hole [24].

To analyse the path integral we have to fix a gauge first. The action (6) looks particularly simple in the "temporal" gauge:

$$e_0^+ = 0, \quad e_0^- = 1, \quad \omega_0 = 0, \quad b_0 = 0\,. \tag{11}$$

Residual gauge freedom can be treated exactly as in the commutative case [16, 22].

Let us introduce the notations:

$$\begin{aligned} q^i &= (e_1^+, e_1^-, \omega_1, b_1)\,, \\ p^i &= (\phi_+, \phi_-, \phi, \psi)\,, \\ \bar{q}^i &= (e_0^+, e_0^-, \omega_0, b_0)\,, \end{aligned} \tag{12}$$

so that the gauge conditions (11) read: $\bar{q}^i - a^i = 0$, where $a^i := (0, 1, 0, 0)$.

Consider a set of the matter fields $\{f^\alpha\}$, where α numbers different components and types of the matter. Spin, statistics or gauge groups play no role in the considerations of this section. If there is an additional gauge symmetry, $\{f^\alpha\}$ should include corresponding ghosts. The only restriction we impose is that these field interact with the "gauge fields" $(e_\mu^a, \omega_\mu, b_\mu) = (q, \bar{q})$, but not with the "Lagrange multipliers" (ϕ_a, ϕ, ψ). The generating functional for the Green functions can be represented as a path integral,

$$\begin{aligned} Z(j, J) = \int \mathcal{D}q\, \mathcal{D}\bar{q}\, \mathcal{D}p\, \mathcal{D}f^\alpha \prod \delta(\bar{q}^i - a^i)\, \mathcal{F} \\ \times \exp\left(iS + iS_{\mathrm{m}}(q, \bar{q}; f^\alpha) + i \int d^2x (j_k q^k + J_k p^k) \right), \end{aligned} \tag{13}$$

where $S_{\mathrm{m}}(q, \bar{q}; f^\alpha)$ is a classical action for the matter fields, $\mathcal{F}$ is the Faddeev-Popov determinant corresponding to our gauge choice. $j = (j_+, j_-, j_3, j_4)$ and $J = (J^+, J^-, J^3, J^4)$ are external sources. One can also introduce sources and/or background fields for the matter.

After imposing the gauge conditions (11) the action (6) becomes

$$S_{\mathrm{g.f.}} = \int d^2x \left[\phi \partial_0 \omega_1 - \psi \partial_0 b_1 + \Lambda \phi e_1^+ + \phi_a \partial_0 e_1^a + \phi_- \omega_1 \right]\,. \tag{14}$$

Since the Moyal product is closed, one can omit stars in integrals of all expressions quadratic in fields (as (14)) provided the fields fall off sufficiently fast at infinity so that one can integrate by parts. This property can be most easily seen by integrating (1).

Now we have to calculate the Faddeev-Popov determinant $\mathcal{F}$. The most advanced technique for construction of the ghost action is based upon the BRST formalism. This formalism has not been yet fully adapted to space-time noncommutative theories (although some steps in this direction have been already done, cf. [26]). Fortunately, since the gauge algebra in our case is of the Yang-Mills type we may use a somewhat simpler Faddeev-Popov prescription [25]. Note, that in the commutative 2D gravities the Faddeev-Popov approach gives correct results even though the gauge group has field-dependent structure functions (cf. Appendix A of [22]). Therefore, we have on the surface defined by the gauge (11)

$$\mathcal{F} = \det\left(\delta\bar{q}^i/\delta\lambda_j\right) = \det(\partial_0)^4 \,, \tag{15}$$

where $\lambda_j = (\alpha^a, \xi, \chi)$ are the gauge parameters (cf. Appendix B).

Since the action (14) is quadratic in the field, and since the Faddeev-Popov determinant (15) is field-independent, it is clear that the integration over p and q in (13) becomes trivial. There are no essential differences to the path integral calculations done in the commutative case [16].

Let us define the effective action W_{m} for the matter fields:

$$W_{\mathrm{m}}(q,\bar{q}) = \frac{1}{i}\ln\int\mathcal{D}f^\alpha e^{iS_{\mathrm{m}}(q,\bar{q};f^\alpha)} \,. \tag{16}$$

We assume that the measure $\mathcal{D}f^\alpha$ does not depend on p^j. The action (16), as it is written here, contains contributions of all matter loops on a background defined by q and $\bar{q}$. One can restrict W_{m} to a finite order of the loop expansion. Then our final result (see eq. (20) below) will be restricted accordingly. By integrating next over the "momentum" variables p^i one obtains the following functional delta-functions:

$$\delta(\partial_0 e_1^+ + J^+) \,, \qquad \delta(\partial_0 e_1^- + \omega_1 + J^-) \,, \tag{17}$$

$$\delta(\partial_0\omega_1 + \Lambda e_1^+ + J^3) \,, \qquad \delta(\partial_0 b_1 - J^4) \,. \tag{18}$$

Because of these delta-functions, integrations over q^i can be also performed exactly. Next we define the mean fields

$$Q = \frac{1}{i}\frac{\delta\ln Z}{\delta j} \,, \qquad P = \frac{1}{i}\frac{\delta\ln Z}{\delta J} \,. \tag{19}$$

and peform the Legendre transform of $\in Z$. The calculations go exactly the same way as in the commutative case. We refer to [16] for details. The effective action reads

$$W(P,Q) = \frac{1}{i}\ln\, Z - \int d^2x(PJ + Qj) = S_{\mathrm{g.f.}}(P,Q) + W_{\mathrm{m}}(Q) \,. \tag{20}$$

This result means that all loop corrections due to the gravity fields disappear[1] if the matter action does not depend on ϕ_a, ϕ and ψ. Of course, such a strong statement is made possible by particular simplicity of the JT model. In more complicated dilaton gravities with matter one has to perform a perturbative expansion already in the commutative case [22].

We have not discussed asymptotic conditions, boundary terms, and other "global" issues. Therefore, we call this kind of statements "local quantum triviality" although locality looses its meaning in noncommutative theories.

4 Conformal anomaly

One usually starts calculations of the conformal anomaly with specifying a classical action for the matter fields. Such an action should of course respect all symmetries of the matterless action (see Appendix B). In the noncommutative limit this action should coincide with a standard action for, say, scalar fields. No such action is known, at least to the present author. The main difficulty is that the diffeomorphism transformations in the noncommutative JT gravity are realised in a very nontrivial way [8]. This fact reflects known problems with constructing covariant coordinate transformations in noncommutative gauge theories (cf. [27]).

Therefore, instead of looking for a classical action for the matter fields, we shall look for an operator which may describe the one-loop corrections in a particular gauge. This idea is inspired by an approach to gravity on noncommutative spaces base on spectral triples [28]. We shall not, however, follow this approach too closely. Our aim is not to construct geometry starting from a spectral triple, but rather to find a meaningful operator starting with geometrical objects of the noncommutative JT gravity.

4.1 Conformal gauge

In this section we work on a space of the Euclidean signature. Most of the formulae derived above remain valid after the substitution $\eta^{ab} = \mathrm{diag}(+1, +1)$, $\varepsilon^{12} = \varepsilon_{12} = 1$. This signature is more convenient to study the heat kernel expansion because, for example, one deals with absolutely convergent integrals.

We need several more simplifying assumptions. First of all, we put

$$b_\mu = 0 \tag{21}$$

since geometric meaning of this field is somewhat obscure. Simple degrees of freedom counting arguments show that now the connection ω_μ can be expressed

[1] The gravity part of the action appears in (20) in the gauge-fixed form. A part of the equations of motion (constraints) is lost and has to be restored by using the Ward identities [22].

through e^a_μ by means of the torsion constraint (10). It is not possible however to solve (10) explicitly unless we impose the conformal gauge condition:

$$e^a_\mu = e^\rho \delta^a_\mu \,. \tag{22}$$

Here e^ρ is a star exponent,

$$e^\rho = 1 + \rho + \frac{1}{2}\rho \star \rho + \frac{1}{6}\rho \star \rho \star \rho + \dots \tag{23}$$

The condition (22) simplifies considerably the combinatorics of all subsequent calculations since now all components of e^a_μ commute with each other. The torsion constraint reads

$$2e^{-\rho} \star (\partial_\mu e^\rho) = -e^{-\rho} \star \hat{\omega}_\mu \star e^\rho - \hat{\omega}_\mu \,, \tag{24}$$

where

$$\hat{\omega}_\mu = \varepsilon_\mu{}^\nu \omega_\nu = \eta_{\mu\rho}\varepsilon^{\rho\nu}\omega_\nu \,. \tag{25}$$

Next we use the Baker-Campbell-Hausdorf (BCH) formula to derive:

$$e^{-\rho} \star (\partial_\mu e^\rho) = \sum_{n=1}^{\infty} \frac{1}{n!}[\dots[[\partial_\mu \rho, \rho], \rho], \dots], \tag{26}$$

$$e^{-\rho} \star \hat{\omega}_\mu \star e^\rho = \sum_{k=0} \frac{1}{k!}[\dots[[\hat{\omega}_\mu, \rho], \rho], \dots]. \tag{27}$$

The nth term in (26) contains $n-1$ commutators, while the kth term in (27) contains k commutators. One can easily prove that

$$\hat{\omega}_\mu = \sum_{n=1}^{\infty} c_n [\dots[[\partial_\mu \rho, \rho], \rho], \dots] \tag{28}$$

(with $n-1$ commutators in the nth term), where $c_1 = -1$ and the subsequent coefficients are given by the recursion:

$$c_n = -\frac{1}{n!} - \frac{1}{2}\sum_{k=1}^{n-1} \frac{c_{n-k}}{k!} \,. \tag{29}$$

In particular, all even-numbered coefficients vanish, $c_{2k} = 0$, and

$$\hat{\omega}_\mu = -\partial_\mu \rho + \frac{1}{12}[[\partial_\mu \rho, \rho], \rho] + \mathcal{O}(\rho^5). \tag{30}$$

The expansion (28) is, by the construction, an expansion in ρ. However, since it contain repeated commutators, it is also an expansion in the noncommutativity parameter θ.

Inspired by (6) one can define the scalar curvature density as

$$\mathcal{R} = \frac{1}{2}\varepsilon^{\mu\nu}\varepsilon_{ab}R^{ab}_{\mu\nu}\,. \tag{31}$$

If $b = 0$,

$$\mathcal{R} = 2\varepsilon^{\mu\nu}\partial_\mu\omega_\nu = 2\partial_\mu\hat{\omega}_\mu\,. \tag{32}$$

In the conformal gauge the metric $G_{\mu\nu} = e^a_\mu \star e^b_\nu \eta_{ab}$ and the volume two-form $E_{\mu\nu} = e^a_\mu \star e^b_\nu \varepsilon_{ab}$ are real.

4.2 Dirac and Laplace operators

Next we like to define noncommutative deformations of the Dirac and Laplace operators. We require that these operators coincide with their commutative counterparts in the limit $\theta \to 0$, and that they are hermitian with respect to some "natural" inner product. This procedure is rather unrigorous, but, as we shall see below, the choice of meaningful deformations is indeed very limited.

We start with the Dirac operator. Let γ^μ be the *flat* space constant γ-matrices:

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\delta^{\mu\nu}\,, \tag{33}$$

i.e. $(\gamma^1)^2 = (\gamma^2)^2 = 1$. Let $\gamma_* = \gamma^1\gamma^2$. This definition is convenient in the conformal gauge.

We have to fix a scalar product in the space of spinors. Let κ_1 and κ_2 be spinorial fields (this means simply two-component complex fields in this context, nothing more). Then

$$\langle\kappa_1,\kappa_2\rangle = \int d^2x\,\kappa_1^\dagger \star e^{2\rho} \star \kappa_2\,. \tag{34}$$

This product is linear, symmetric, and positive as long as both e^ρ and $e^{-\rho}$ are well defined. We can even get rid of $e^{2\rho}$ by changing the variables to the densitised fields $\tilde{\kappa} = e^\rho\kappa$. Then

$$\langle\kappa_1,\kappa_2\rangle = \langle\tilde{\kappa}_1,\tilde{\kappa}_2\rangle_0 = \int d^2x\,\tilde{\kappa}_1^\dagger\tilde{\kappa}_2\,. \tag{35}$$

Note, that $\langle\ ,\ \rangle_0$ does not contain star at all. By a similar change of the variables one can move $e^{2\alpha\rho}$, with α being a real constant, in front of $\kappa_1^\dagger$ in (34) so that the scalar product becomes:

$$\widetilde{\langle\kappa_1,\kappa_2\rangle} = \int d^2x\,e^{2\alpha\rho} \star \kappa_1^\dagger \star e^{2(1-\alpha)\rho} \star \kappa_2\,. \tag{36}$$

This product seems to be the most general meaningful deformation of the standard commutative inner product. We see, that although there exists different

choices of the inner product, they all are related by a change of variables. We choose $\alpha = 0$, i.e. the product defined in (34).

Let us define the Dirac operator by the equation

$$\widehat{D} = i\gamma^\mu e^{-\rho} \star \left(\partial_\mu + \frac{1}{2}\omega_\mu \gamma_* \right) . \tag{37}$$

This operator is fixed by its' commutative counterpart up to the order in which we write $e^{-\rho}$ and ω. This order is then uniquely defined by the requirement that $\widehat{D}$ is hermitian[2] with respect to the inner product (34):

$$\langle \widehat{D} \star \kappa_1, \kappa_2 \rangle = \langle \kappa_1, \widehat{D} \star \kappa_2 \rangle \tag{38}$$

We stress that all multiplications are the Moyal star products. There is no additional ambiguity related to the choice of the Dirac operator. Let us remind that we have fixed $b_\mu = 0$. Otherwise, b_μ should have also appeared in (37).

Note, that the operator (37) does not define any spectral triple (cf. sec. 4.4 of [5]). The reason is that the commutator $[\widehat{D}, f] = i\gamma^\mu [e^{-\rho}, f]\partial_\mu + \dots$, where f is a function, is not a bounded operator because of the presence of the "first-order" term proportional to ∂_μ. This difficulty is hardly possible to avoid if one likes to identify the leading symbol of the Dirac operator (i.e. the term appearing in front of ∂_μ) with a zweibein of a noncommutative gravity theory on the Moyal plane (of which the noncommutative JT gravity considered above is an example).

By using the torsion constraint (24) one can prove an analog of the Lichnerowicz formula

$$\widehat{D}^2 = \Delta_{\text{Spin}} + \frac{1}{2} e^{-2\rho} \star \epsilon^{\mu\nu} (\partial_\mu \omega_\nu) + \frac{1}{4} e^{-2\rho} \star \gamma_* \epsilon^{\mu\nu} \omega_\mu \star \omega_\nu \, , \tag{39}$$

where the spinor Laplacian reads

$$\Delta_{\text{Spin}} = -e^{-2\rho} \star \left(\partial_\mu + \frac{1}{2}\gamma_* \omega_\mu \right)^2 . \tag{40}$$

Note, that the 2nd and 3rd terms on the right hand side of (39) are not necessarily real. The reason is that left multiplication by a real function is not a hermitian operation with respect to the inner product (34) if this function does not commute with $e^{2\rho}$.

Basing on (39) we may conjecture that there exist generalisations of the Dirac and Laplace operator such that the following formula holds for generic e_μ^a and generic torsionless connection (including b_μ):

$$\widehat{D}^2 = \Delta_{\text{Spin}} + \frac{1}{8}\{e_a^\mu, e_b^\nu\} \star R_{\mu\nu}^{ab} + \frac{1}{8}\{e_a^\mu, e_b^\nu\} \star \epsilon^{ab} \eta_{cd} R_{\mu\nu}^{cd} \gamma_* \, . \tag{41}$$

[2]An operator satisfying (38) is also called symmetric or formally self-adjoined. A self-adjoint operator must satisfy one additional requirement regarding its domain. We shall not consider this requirement, as well as we shall ignore such issues as completeness of the Hilbert spaces etc.

The second term on the right hand side of (41) is a rather straightforward extension of the $R/4$ term appearing in commutative theories[3]. The third term is a new feature of noncommutative theories.

Let us now turn to scalar fields. It is natural to assume that in the conformal gauge massless minimally coupled scalar fields decouple from the geometry (as in the commutative case), so that the action reads

$$S_{\mathrm{m}} = \frac{1}{2} \int d^2 x (\partial_\mu f^\dagger)(\partial_\mu f) \tag{42}$$

Now we have to choose an inner product in the space of the scalar fields. Let us fix it to be

$$\langle f_1, f_2 \rangle = \int d^2 x \ f_1^\dagger \star e^{2\rho} \star f_2 \tag{43}$$

in full analogy with the spinor case. We can change the variables, $\tilde{f} = e^\rho f$, so that we obtain a "trivial" inner product

$$\langle \tilde{f}_1, \tilde{f}_2 \rangle_0 = \int d^2 x \ \tilde{f}_1^\dagger \star \tilde{f}_2 = \int d^2 x \ \tilde{f}_1^\dagger \tilde{f}_2 \tag{44}$$

for the new fields $\tilde{f}$.Then

$$S_{\mathrm{m}} = \frac{1}{2} \int d^2 x \tilde{f}^\dagger \star \Delta \star \tilde{f}\,, \tag{45}$$

where

$$\Delta = -e^{-\rho} \star \partial_\mu^2 e^{-\rho}. \tag{46}$$

Next we integrate (formally) over $\tilde{f}$ (cf. (16)) with the measure $\mathcal{D}f^\alpha = \mathcal{D}\tilde{f}^\dagger \mathcal{D}\tilde{f}$ to obtain the effective action

$$W_{\mathrm{m}} = \ln \det \Delta\,. \tag{47}$$

This expression is, of course, divergent and has to be regularised.

4.3 Heat kernel and the anomaly

Actual calculations of the effective action will be done in the scalar case only (this case is already complicated enough). We use the zeta function and heat kernel techniques (see [29] for a recent review) to regularise the determinant (47). The heat kernel expansion on (flat) Moyal spaces was studied in [30, 5]. We start with several definitions. Let h be a smooth function, and t be a positive real number.

[3]With the help of the identity $\varepsilon^{ab}\varepsilon_{cd} = \delta^a_c \delta^b_d - \delta^a_d \delta^b_c$ one can relate this term to the scalar curvature density (31).

Then the heat kernel (or, in a more precise terminology, the heat trace) is defined as

$$K(h,t,\Delta) = \mathrm{Tr}_{L^2}\left(h \star \exp(-t\Delta)\right),. \tag{48}$$

The L^2 space is defined with respect to the inner product (44). Here noncommutativity plays no role, and L^2 consists of all square integrable functions on $\mathbb{R}^2$. We shall assume that Δ is a positive operator, so that we can define the zeta function:

$$\zeta(h,s,\Delta) = \mathrm{Tr}_{L^2}\left(h \star \Delta^{-s}\right). \tag{49}$$

The zeta function is related to the heat kernel by the integral transformation

$$\zeta(h,s,\Delta) = \Gamma(s)^{-1}\int_0^\infty dt\, t^{s-1} K(h,t,\Delta). \tag{50}$$

Note, that the existence of the heat kernel and of the zeta function for the operator (46) has not been rigorously stated so far in the noncommutative case. Below we shall present some arguments showing that these object do exist at the level of rigour accepted in physics. We shall also demonstrate that, at least in the sense of formal power series in ρ, there is an asymptotic expansion as $t \to +0$:

$$K(h,t,\Delta) \simeq \sum_{k=0}^{\infty} t^{k-1} a_{2k}(h,\Delta). \tag{51}$$

As in the commutative case, there are no terms with fractional powers of t.

The heat kernel coefficients a_{2k} are related to the residues of $\zeta\Gamma$.

$$a_{2k}(h,\Delta) = \mathrm{Res}_{s=1-k}\left(\Gamma(s)\zeta(h,s,\Delta)\right). \tag{52}$$

In particular,

$$a_2(h,\Delta) = \zeta(h,0,\Delta). \tag{53}$$

It has been proposed in [31] to use the zeta function to regularise the effective action:

$$W_s = -\mu^2 \int_0^\infty dt\, t^{s-1} K(t,\Delta) = \mu^2 \Gamma(s)\zeta(s,\Delta). \tag{54}$$

Here μ is a constant of the dimension of mass introduced to keep proper dimension of the effective action. Spectral functions without the first argument correspond to $h=1$, i.e. $K(t,\Delta) := K(1,t,\Delta)$, $\zeta(s,\Delta) := \zeta(1,s,\Delta)$. The regularization is removed in the limit $s \to 0$. At $s=0$ the effective action has a simple pole:

$$W_s = -\left(\frac{1}{s} - \gamma_E + \ln\mu^2\right)\zeta(0,\Delta) - \zeta'(0,\Delta), \tag{55}$$

where γ_E is the Euler constant, prime denotes differentiation with respect to s. The second term on the right hand side of (55) is nothing else than the Ray-Singer definition of the functional determinant [32].

The pole terms should be removed by the renormalization. The remaining part of W_s is the renormalised effective action:

$$W_{\rm m}^{\rm ren} = -\ln(\mu^2)\zeta(0,\Delta) - \zeta'(0,\Delta)\,. \tag{56}$$

The constant μ^2, which is arbitrary so far, describes the renormalization ambiguity and has to be fixed by a normalisation condition.

Note, that the operator Δ and, consequently, the effective action (56) depend on the conformal field ρ only. Therefore, we can use the conformal anomaly to calculate $W_{\rm m}^{\rm ren}$. Let us rescale $\rho \to \alpha\rho$, where α is a real parameter between 0 and 1. Then we introduce

$$\Delta_{[\alpha]} = -e^{-\alpha\rho} \star \partial^2 e^{-\alpha\rho} \tag{57}$$

and the corresponding effective action $W_{\rm m}^{\rm ren}(\alpha)$. We may write

$$\begin{aligned} &\frac{d}{d\alpha}\zeta(s,\Delta_{[\alpha]}) = -s\mathrm{Tr}\left(\frac{d}{d\alpha}\Delta_{[\alpha]} \star \Delta_{[\alpha]}^{-s-1}\right) = 2s\mathrm{Tr}\left(\rho \star \Delta_{[\alpha]}^{-s}\right) \\ &= 2s\zeta(\rho,s,\Delta_{[\alpha]})\,. \end{aligned} \tag{58}$$

By combining (58) with (56) and (53) we obtain:

$$\frac{d}{d\alpha}W_{\rm m}^{\rm ren}(\alpha) = -2a_2(\rho,\Delta_{[\alpha]}). \tag{59}$$

This quantity describes conformal non-invariance of the effective action (conformal anomaly).

We arrive at the problem of calculating $a_2(h,\Delta)$. If we set $h=1$ this calculation should give us the divergent part of the effective action (55). For $h=\rho$ and $\rho \to \alpha\rho$ insider the operator the same calculation also defines the conformal variation (59). We have to evaluate the small t asymptotics of the heat trace for the noncommutative Laplacian Δ. Recently, it was realised [30, 5] that the noncommutativity by itself is not a big problem. The problem is that the star multiplication by a functions appears in a combination with the highest derivatives of the operator (i.e. the star multiplication enters the leading symbol of the operator). Heat trace calculations with operators in which highest derivatives are multiplied by an (almost) arbitrary matrix valued function appeared recently in the context of a matrix generalisation of general relativity [33], and long ago in the context of the bosonisation of QCD [34]. Our calculations (see below) share some similarities with the above mentioned papers, though our case is even more complicated.

The general strategy adopted here is taken from [30]. To evaluate (48) we sandwich the expression under the trace between plane waves and then integrate over x and k:

$$K(h,t,\Delta) = \int d^2x \int \frac{d^2k}{(2\pi)^2} e^{-ikx} \star h(x) \star e^{-t\Delta} \star e^{ikx}\,. \tag{60}$$

Note, that the plane wave basis is orthormal with respect to the inner product (44). Next we push e^{ikx} to the left to obtain:

$$K(h,t,\Delta) = \int d^2x \int \frac{d^2k}{(2\pi)^2}\, h(x) \\ \star \exp\left(-tk^2 e^{-2\rho} + te^{-\rho} \star \partial^2 e^{-\rho} + 2ie^{-\rho} \star (k\partial)e^{-\rho}\right), \quad (61)$$

where $(k\partial) := k^\mu \partial_\mu$, $k^2 := k^\mu k^\nu \delta_{\mu\nu}$. The most important observation needed to derive (61) was made in [30]. Under the integral over x the plane wave e^{ikx} can be pushed through the Moyal star without any modifications of the latter[4], so that the only effect of this operation is the replacement $\partial^2 \to (\partial + ik)^2$ in Δ.

Let $e^{-2\rho}$ be bounded from the below by a positive constant c, $0 < c \le e^{-2\rho}$. Then, for large k the integrand in (61) falls off as e^{-tck^2}. Therefore, the integral over k converges for all x. For a sufficiently good localised function $h(x)$ the integral over x should also exist. This is the physical argument in favour of the existence of the heat trace we have announced above.

Actually, we are interested in the $t \to 0$ asymptotics of the heat trace only. To evaluate these asymptotics one has to isolate $\exp(-tk^2 e^{-2\rho})$, expand the rest, and integrate over k. The following integrals will be useful,

$$\int \frac{d^2k}{(2\pi)^2} e^{-ak^2} k^{2n} = \frac{n!}{4\pi a^{n+1}}\,, \quad (62)$$

$$\int \frac{d^2k}{(2\pi)^2} e^{-ak^2} k_\mu k_\nu k^{2n} = \frac{1}{2}\delta_{\mu\nu} \frac{(n+1)!}{4\pi a^{n+2}}\,, \quad (63)$$

where $a = e^{-2\rho}t$, all exponentials and powers are defined with the Moyal star product (for example, $a \star a^{-1} = 1$). The formulae (62) and (63) are obvious in the commutative case, but are less trivial in the noncommutative one. To obtain (62) and (63) one has to represent $a = t + (e^{-2\rho} - 1)t$, keep e^{-tk^2} and expand the rest into a power series in ρ. Then one integrates over k and sums up the series.

Before expanding the exponent in (61) it is useful to estimate the power of t in each of the individual terms. In terms of the rescaled variable $\tilde{k} = t^{1/2}k$ the expression in the exponent (61) reads:

$$-\tilde{k}^2 e^{-2\rho} + t^{1/2} 2ie^{-\rho} \star (\tilde{k}\partial)e^{-\rho} + te^{-\rho} \star \partial^2 e^{-\rho}\,. \quad (64)$$

The integration measure $d^2k = t^{-1} d^2\tilde{k}$ produces an overall factor of t^{-1}. The second term in (64) contributes $t^{1/2}$, and the third contributes t to the expansion. It is also clear that all half-integer powers of t vanish after integration over the momenta.

[4]This means that in the derivation of this formula the derivatives appearing in the Moyal product (1) can be safely ignored.

The coefficient $a_0(h,\Delta)$ is relatively easy to obtain:

$$a_0(h,\Delta) = \int d^2x h \star \int \frac{d^2\tilde{k}}{(2\pi)^2} \exp(-e^{-2\rho}\tilde{k}^2) = \frac{1}{4\pi}\int d^2x h \star e^{2\rho}\,. \qquad (65)$$

This coefficient is an obvious generalisation of corresponding commutative expression.

Calculations of a_2 are much more involved. One has to keep the terms which are either second order in the second term in (64), or linear in the third term. The first term should be taken into account exactly. After long but rather elementary calculations we obtain:

$$\begin{aligned} a_2(h,\Delta) &= \frac{1}{4\pi}\int d^2x\, h(x) \\ &\star \Bigg(\sum_{n=0}^{\infty} \frac{1}{n+1} e^{2(n+1)\rho} \star \underbrace{[e^{-2\rho},[e^{-2\rho},[\dots,e^{-\rho}\star\partial^2 e^{-\rho}]]]}_{n \text{ commutators}} - \qquad (66) \\ &- \sum_{m,n=0}^{\infty} \frac{2(n+m+1)!e^{2(n+m+2)\rho}}{n!(m+1)!(n+m+2)} \star \underbrace{[e^{-2\rho},[e^{-2\rho},[\dots,e^{-\rho}\star\partial_\mu e^{-\rho}]]]}_{n \text{ commutators}} \\ &\star \underbrace{[e^{-2\rho},[e^{-2\rho},[\dots,e^{-\rho}\star\partial_\mu e^{-\rho}]]]}_{m \text{ commutators}} \Bigg) \end{aligned}$$

It seems that the best one can do with (66) is to extract several leading terms of an expansion in ρ. Fortunately, only a finite number of terms in (66) contribute to any finite order of this expansion.

$$\begin{aligned} a_2(h,\Delta) = \frac{1}{4\pi}\int d^2x\, h(x) \star \Bigg(&-\frac{1}{3}\partial^2\rho + \frac{1}{30}[[\rho,(\partial_\mu\rho)],(\partial_\mu\rho)] \\ &+ \frac{7}{90}\partial_\mu[[(\partial_\mu\rho),\rho],\rho] + \mathcal{O}(\rho^4) \Bigg)\,. \qquad (67) \end{aligned}$$

One may expect that a_2 is given by an analog of the "commutative" expression:

$$a_2(h,\Delta)_{\text{com}} = \frac{1}{4\pi}\int d^2x \frac{1}{6} h(x)\mathcal{R}\,, \qquad (68)$$

where $\mathcal{R}$ is the scalar curvature density (31). With the help of (32) and (30) one can easily check that (68) remains true in the noncommutative case in the orders ρ and ρ^2, but is violated in the order ρ^3. The origin of this deviation is yet unclear.

Next we notice that $a_2(1,\Delta)$ is given by surface terms at least up to the order ρ^3. Therefore, there are no local divergences in (55), and no local counterterms are needed for the renormalization.

Now we can integrate (59) to obtain an analog of the Polyakov action [18] (see also [35]). It is natural to assume that $W_{\rm m}^{\rm ren}(\alpha = 0) = 0$. Then

$$W_{\rm m}^{\rm ren} = -\frac{1}{4\pi} \int d^2x \left(-\frac{1}{3} \rho \star \partial_\mu^2 \rho + \frac{1}{45} [\rho, \partial_\mu \rho] \star [\rho, \partial_\mu \rho] + \mathcal{O}(\rho^5) \right) . \tag{69}$$

The first term on the right hand side is just the standard anomaly induced action. The second term appears due to the noncommutativity and vanishes in the commutative limit $\theta \to 0$.

We have to admit that our choice for the Laplacian is not unique. There can be other NC Laplacians which may be more relevant for physical or mathematical applications. However, main technical tools developed in this paper should remain applicable for analysing the heat kernel expansion for that other Laplacians, perhaps at the expense of some technical complications.

5 Conclusions

In this paper we have studied quantisation of noncommutative gravity theories in two dimensions. We started with the path integral in the noncommutative JT model and demonstrated that all gravitational degrees of freedom can be integrated out exactly even in the presence of matter fields (with some minor restrictions on the interaction between matter and gravity). The resulting quantum effective action coincides with the classical action for gravity plus an effective action for the matter calculated as if the gravity fields were classical. Then we studied the matter effective action (restricting ourselves to the conformal gauge for simplicity). We have found several natural differential operators which may describe quantum matter fluctuations and studied their properties. In particular, a noncommutative analog of the Lichnerowicz formula has been derived. Then we turned to the scalar Laplacian, evaluated two leading terms of the heat kernel expansion, calculated conformal anomaly (as an expansion in the conformal field ρ), and found a noncommutative analog of the Polyakov action.

Note that the noncommutative JT gravity is the first noncommutative (Moyal) quantum gravity model there one can go this far. Our main message is, therefore, that noncommutative gravities can indeed be successfully quantised.

The results obtained here can be improved in many respects. Let us outline just a few directions of possible future developments.

- It is interesting to construct noncommutative deformations of two-dimensional dilaton gravities other than JT. In this more general context the gauge theory formulation (2) is not applicable.

- In the matter sector we have only constructed a natural fluctuation operator in the conformal gauge. It is important to check whether this operator

corresponds to an action which possesses all necessary symmetries. With such an action at hand, one may study many interesting phenomena as the black hole formation, scattering on black holes (perhaps, with bound state formation [36]), etc.

- One definitely has to pay more attention to a careful definition of the path integral, especially in the presence of the space-time noncommutativity (cf. recent work [37]).

- One has to undertake a more rigorous study of the heat kernel and of the zeta function for the operators on "curved" Moyal plane (or Moyal torus).

- Although the noncommutative JT model [8] has a proper number of gauge symmetries, including Lorentz boosts and diffeomorphisms, the restriction to a constant noncommutativity parameter θ does not look very natural since it implicitly selects a coordinate system. It is an interesting problem to construct a gravity theory with the Kontsevich star [38] instead of the Moyal one.

Acknowledgements

I am grateful to Daniel Grumiller and Max Kreuzer for many helpful discussions. This work has been supported in part by the DFG project BO 1112/12-1.

A Notations and conventions

Our sign conventions are mostly taken from [15]. We use the tensor $\eta^{ab} = \eta_{ab} = \mathrm{diag}(+1, -1)$ to move indices up and down. The Levi-Civita tensor is defined by $\varepsilon^{01} = -1$, so that the following relations hold

$$\varepsilon^{10} = \varepsilon_{01} = 1, \qquad \varepsilon^{0}{}_{1} = \varepsilon^{1}{}_{0} = -\varepsilon_{0}{}^{1} = -\varepsilon_{1}{}^{0} = 1\,. \tag{70}$$

These relations are valid for both ε^{ab} and $\varepsilon^{\mu\nu}$. Note, that $\varepsilon^{\mu\nu}$ is always used with both indices up.

We also use the light-cone basis in which

$$\begin{aligned} &\eta_{+-} = \eta_{-+} = \eta^{+-} = 1, \qquad \eta^{++} = \eta_{++} = \eta^{--} = \eta_{--} = 0\,, \\ &\varepsilon^{+}{}_{+} = -\varepsilon^{-}{}_{-} = 1, \qquad \varepsilon_{-+} = -\varepsilon_{+-} = \varepsilon^{+-} = -\varepsilon^{-+} = 1. \end{aligned} \tag{71}$$

In sec. 4 we use the Euclidean signature, so that

$$\eta^{11} = \eta^{22} = 1, \qquad \varepsilon^{12} = -\varepsilon^{21} = 1\,. \tag{72}$$

B Symmetry transformations

Here we present an explicit form of the gauge symmetries of the non-commutative JT model [8][5].

Translations:

$$\begin{aligned}
&\delta_\alpha e^a_\mu = \partial_\mu \alpha^a + \frac{1}{2}\varepsilon^a{}_b\{\omega_\mu, \alpha^b\} + \frac{i}{2}[b_\mu, \alpha^a]\,,\\
&\delta_\alpha \phi^a = -\Lambda\left(\frac{1}{2}\varepsilon^a{}_b\{\phi, \alpha^b\} + \frac{i}{2}[\psi, \alpha^a]\right)\,,\\
&\delta_\alpha \omega_\mu = \frac{\Lambda}{2}\varepsilon_{ab}\{e^a_\mu, \alpha^b\}, \qquad \delta_\alpha \phi = -\frac{1}{2}\varepsilon_{ab}\{\phi^a, \alpha^b\},\\
&\delta_\alpha b_\mu = \frac{i\Lambda}{2}\eta_{ab}[e^a_\mu, \alpha^b], \qquad \delta_\alpha \psi = -\frac{i}{2}\eta_{ab}[\phi^a, \alpha^b].
\end{aligned} \tag{73}$$

Boosts:

$$\begin{aligned}
&\delta_\xi e^a_\mu = -\frac{1}{2}\varepsilon^a{}_b\{e^b_\mu, \xi\}, \qquad \delta_\xi \phi^a = -\frac{1}{2}\varepsilon^a{}_b\{\phi^b, \xi\},\\
&\delta_\xi \omega_\mu = \partial_\mu \xi + \frac{i}{2}[b_\mu, \xi], \qquad \delta_\xi \phi = \frac{i}{2}[\psi, \xi],\\
&\delta_\xi b_\mu = -\frac{i}{2}[\omega_\mu, \xi], \qquad \delta_\xi \psi = -\frac{i}{2}[\phi, \xi].
\end{aligned} \tag{74}$$

$U(1)$ gauge symmetry:

$$\begin{aligned}
&\delta_\chi e^a_\mu = \frac{i}{2}[e^a_\mu, \chi], \qquad \delta_\chi \phi^a = \frac{i}{2}[\phi^a, \chi],\\
&\delta_\chi \omega_\mu = \frac{i}{2}[\omega_\mu, \chi], \qquad \delta_\chi \phi = \frac{i}{2}[\phi, \chi],\\
&\delta_\chi b_\mu = \partial_\mu \chi + \frac{i}{2}[b_\mu, \chi], \qquad \delta_\chi \psi = \frac{i}{2}[\psi, \chi].
\end{aligned} \tag{75}$$

References

[1] Douglas M. R. and Nekrasov N. A., Noncommutative field theory, Rev. Mod. Phys. **73** (2001) 977-1029 [arXiv:hep-th/0106048];

Szabo R. J., Quantum field theory on noncommutative spaces, Phys. Rept. **378** (2003) 207-299 [arXiv:hep-th/0109162].

[2] M. Buric and J. Madore, Noncommutative 2-dimensional models of gravity, preprint ESI 1467 (Vienna, 2004), arXiv:hep-th/0406232.

[5] Here we correct a few misprints in corresponding formulae of Ref. [8]. The present author is grateful to Sergio Cacciatori and Luca Martucci for correspondence regarding this point.

[3] A. H. Chamseddine and A. Connes, The spectral action principle, Commun. Math. Phys. **186** (1997) 731 [arXiv:hep-th/9606001].

[4] A. H. Chamseddine, Noncommutative gravity, Annales Henri Poincare **4S2** (2003) S881 [arXiv:hep-th/0301112].

[5] V. Gayral and B. Iochum, The spectral action for Moyal planes, arXiv:hep-th/0402147.

[6] A. H. Chamseddine, Complexified gravity in noncommutative spaces, Commun. Math. Phys. **218** (2001) 283 [arXiv:hep-th/0005222].

[7] J. W. Moffat, Noncommutative quantum gravity, Phys. Lett. B **491** (2000) 345 [arXiv:hep-th/0007181].

[8] S. Cacciatori, A. H. Chamseddine, D. Klemm, L. Martucci, W. A. Sabra and D. Zanon, Noncommutative gravity in two dimensions, Class. Quant. Grav. **19** (2002) 4029 [arXiv:hep-th/0203038].

[9] M. Banados, O. Chandia, N. Grandi, F. A. Schaposnik and G. A. Silva, Three-dimensional noncommutative gravity, Phys. Rev. D **64** (2001) 084012 [arXiv:hep-th/0104264].

[10] A. H. Chamseddine, Invariant actions for noncommutative gravity, J. Math. Phys. **44** (2003) 2534 [arXiv:hep-th/0202137];

M. A. Cardella and D. Zanon, Noncommutative deformation of four dimensional Einstein gravity, Class. Quant. Grav. **20** (2003) L95 [arXiv:hep-th/0212071].

[11] H. Garcia-Compean, O. Obregon, C. Ramirez and M. Sabido, Noncommutative self-dual gravity, Phys. Rev. D **68** (2003) 044015 [arXiv:hep-th/0302180];

H. Garcia-Compean, O. Obregon and C. Ramirez, Noncommutative topological half-flat gravity, arXiv:hep-th/0402168.

[12] J. W. Moffat, Perturbative noncommutative quantum gravity, Phys. Lett. B **493** (2000) 142 [arXiv:hep-th/0008089].

[13] H. Quevedo and J. G. Tafoya, Towards the deformation quantization of linearized gravity, arXiv:gr-qc/0401088.

[14] F. Ardalan, H. Arfaei, M. R. Garousi and A. Ghodsi, Gravity on noncommutative D-branes, Int. J. Mod. Phys. A **18** (2003) 1051 [arXiv:hep-th/0204117].

[15] D. Grumiller, W. Kummer and D. V. Vassilevich, Dilaton gravity in two dimensions, Phys. Rept. **369** (2002) 327 [arXiv:hep-th/0204253].

[16] W. Kummer, H. Liebl and D. V. Vassilevich, Exact path integral quantization of generic 2-D dilaton gravity, Nucl. Phys. B **493** (1997) 491 [arXiv:gr-qc/9612012].

[17] C. Teitelboim, Gravitation And Hamiltonian Structure In Two Space-Time Dimensions, Phys. Lett. B **126** (1983) 41; The Hamiltonian Structure of Two-Dimensional Space-Time and its Relation with the Conformal Anomaly, in Quantum Theory Of Gravity, p. 327-344, S.Christensen (ed.), Adam Hilgar, Bristol;

R. Jackiw, Liouville Field Theory: A Two-Dimensional Model For Gravity? in Quantum Theory Of Gravity, p. 403-420, S.Christensen (ed.), Adam Hilgar, Bristol; Lower Dimensional Gravity, Nucl. Phys. B **252** (1985) 343.

[18] A. M. Polyakov, Quantum geometry of bosonic strings, Phys. Lett. **B103** (1981) 207–210.

[19] F. A. Berezin and M. A. Shubin, *The Schrödinger Equation*, Kluwer, Dordrecht, 1991.

[20] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer, Deformation Theory And Quantization. 1. Deformations Of Symplectic Structures, Annals Phys. **111** (1978) 61-110.

[21] W. Kummer, H. Liebl and D. V. Vassilevich, Non-perturbative path integral of 2d dilaton gravity and two-loop effects from scalar matter, Nucl. Phys. B **513** (1998) 723 [arXiv:hep-th/9707115].

[22] W. Kummer, H. Liebl and D. V. Vassilevich, Integrating geometry in general 2D dilaton gravity with matter, Nucl. Phys. B **544** (1999) 403 [arXiv:hep-th/9809168].

[23] L. Bergamin, D. Grumiller and W. Kummer, Quantization of 2D dilaton supergravity with matter, arXiv:hep-th/0404004.

[24] D. Grumiller, W. Kummer and D. V. Vassilevich, Positive specific heat of the quantum corrected dilaton black hole, J. High Energy Phys. **0307** (2003) 009 [arXiv:hep-th/0305036].

[25] L. D. Faddeev and V. N. Popov, Feynman Diagrams For The Yang-Mills Field, Phys. Lett. B **25** (1967) 29.

[26] G. Barnich, F. Brandt and M. Grigoriev, Local BRST cohomology and Seiberg-Witten maps in noncommutative Yang-Mills theory, Nucl. Phys. B **677** (2004) 503 [arXiv:hep-th/0308092].

[27] R. Jackiw and S. Y. Pi, Covariant coordinate transformations on noncommutative space, Phys. Rev. Lett. **88** (2002) 111603 [arXiv:hep-th/0111122].

[28] A. H. Chamseddine, G. Felder and J. Fröhlich, Gravity in noncommutative geometry, Commun. Math. Phys. **155** (1993) 205 [arXiv:hep-th/9209044];

A. H. Chamseddine, J. Fröhlich and O. Grandjean, The Gravitational sector in the Connes-Lott formulation of the standard model, J. Math. Phys. **36** (1995) 6255 [arXiv:hep-th/9503093];

A. Connes, Noncommutative Geometry And Reality, J. Math. Phys. **36** (1995) 6194.

[29] D. V. Vassilevich, Heat kernel expansion: User's manual, Phys. Rept. **388** (2003) 279 [arXiv:hep-th/0306138].

[30] D. V. Vassilevich, Non-commutative heat kernel, Lett. Math. Phys. **67** (2004) 185 [arXiv:hep-th/0310144].

[31] J. S. Dowker and R. Critchley, Effective Lagrangian And Energy Momentum Tensor In De Sitter Space, Phys. Rev. D **13** (1976) 3224;

S. W. Hawking, Zeta Function Regularization Of Path Integrals In Curved Space-Time, Commun. Math. Phys. **55** (1977) 133.

[32] D. Ray and I. M. Singer, R-torsion and the Laplacian on Riemannian manifolds, Adv. Math. **7** (1971) 145-210.

[33] I. G. Avramidi, Gauged Gravity via Spectral Asymptotics of non-Laplace type Operators, JHEP **0407** (2004) 030 [arXiv:hep-th/0406026].

[34] A. A. Andrianov, V. A. Andrianov, V. Y. Novozhilov and Y. V. Novozhilov, Joint Chiral And Conformal Bosonization In QCD And The Linear Sigma Model, Phys. Lett. B **186** (1987) 401.

[35] M. Lüscher, K. Symanzik, and P. Weisz, Anomalies of the free loop wave equation in the WKB approximation, Nucl. Phys. **B173** (1980) 365.

[36] D. V. Vassilevich and A. Yurov, Space-time non-commutativity tends to create bound states, Phys. Rev. D **69** (2004) 105006 [arXiv:hep-th/0311214].

[37] K. Fujikawa, Path Integral for Space-time Noncommutative Field Theory, Phys. Rev. D **70** (2004) 085006 [arXiv:hep-th/0406128].

[38] M. Kontsevich, Deformation quantization of Poisson manifolds, I, Lett. Math. Phys. **66** (2003) 157 [arXiv:q-alg/9709040];

A. S. Cattaneo and G. Felder, On the globalization of Kontsevich's star product and the perturbative Poisson sigma model, Prog. Theor. Phys. Suppl. **144** (2001) 38 [arXiv:hep-th/0111028].